Some Euclidean Geometry for Mathematical Olympiads

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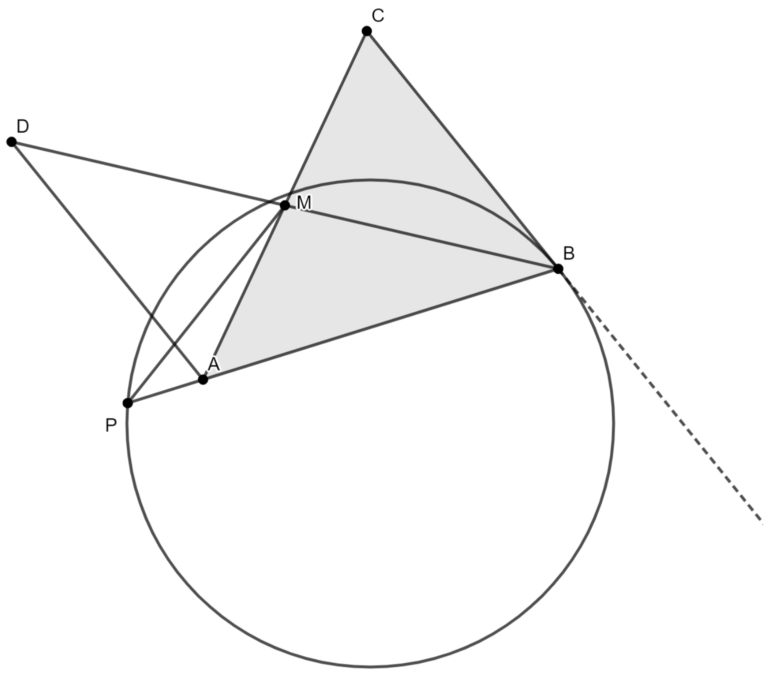
1 Example Problem 1

This is **problem 1** from *BMO2 2018*.

> Consider triangle ABC. The midpoint of AC is M. The circle tangent to BC at B and passing through M meets the line AB again at P. Prove that $AB \times BP = 2BM^2$.

Solution. We see this product that we are required to prove. This resembles ratios of lengths. I think similar triangles should be screaming out at us.



$$AB \times BP = 2BM^2 \implies \frac{AB}{2BM} = \frac{BM}{BP}$$

Point D is the line BM extended out such that M is the midpoint of BD - this gives us the '$2BM$'.

Clearly, all that is required is to show $\triangle MPB \sim \triangle ABD$. By constructing point D, we have bisecting diagonals (since $AM = MC$ also) so $ABCD$ is parallelogram.

By alternate segment theorem, $\angle DBC = \angle MPB$. Also, $AD \parallel BC$ so $\angle DBC = \angle BDA$.

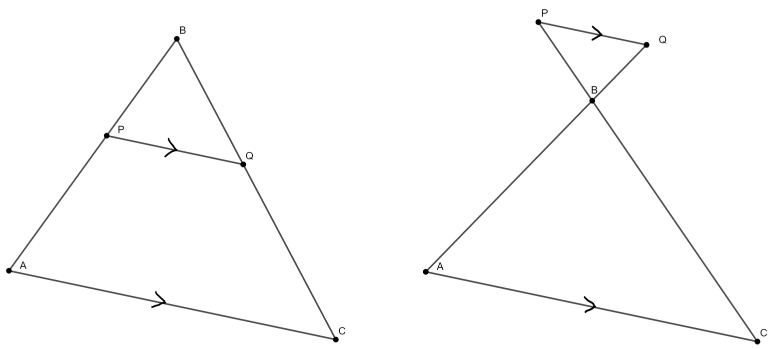
$$\therefore\ \angle DBA = \angle MPB \implies \triangle MPB \sim \triangle ABD \quad (\angle PBD \text{ common})$$

Then, the result follows. □

2 Some useful theorems

All of these are indispensable for us to know, and they are very cool indeed. With these theorems or ‘tools’, we can be in good stead for tackling a huge number of geometry problems, including within olympiads.

2.1 Intercept Theorem



$$\textbf{RTP: } \frac{BP}{PA} = \frac{BQ}{QC}$$

Let’s consider areas.

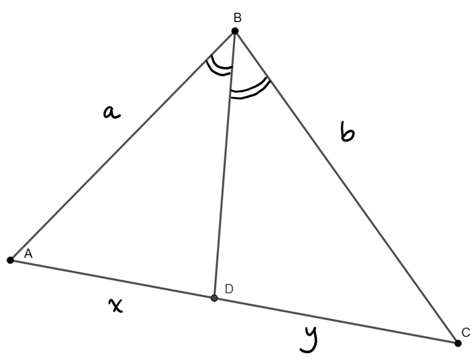
$$[BPQ] = \frac{1}{2}h \cdot BP, \ [AQP] = \frac{1}{2}h \cdot PA \implies \frac{[BPQ]}{[AQP]} = \frac{BP}{PA}$$

$$[BPQ] = \frac{1}{2}h' \cdot BQ, \ [PCQ] = \frac{1}{2}h' \cdot QC \implies \frac{[BPQ]}{[CQP]} = \frac{BQ}{QC}$$

$$\text{But } [CQP] = [AQP] \ \therefore \ \boxed{\frac{BP}{PA} = \frac{BQ}{QC}}$$

. □

2.2 Angle Bisector Theorem



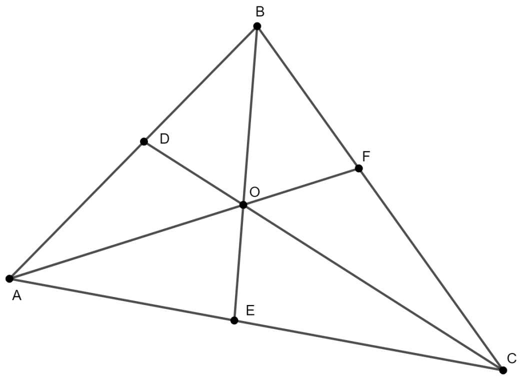
By sine rule,

$$\frac{a}{\sin \angle ADB} = \frac{x}{\sin \angle DBA}$$
$$\frac{b}{\sin(180^\circ - \angle ADB)} = \frac{y}{\sin \angle DBA} \implies \boxed{\frac{a}{b} = \frac{x}{y}}$$

□

2.3 Ceva's Theorem

This theorem addresses **concurrency**.



Considering areas,

$$\frac{AE}{EC} = \frac{[AEB]}{[CBE]} = \frac{[AEO]}{[COE]} = \frac{[AEB]-[AEO]}{[ABE]-[COE]} = \frac{AOB}{BOC}$$ [1]

Similarly,

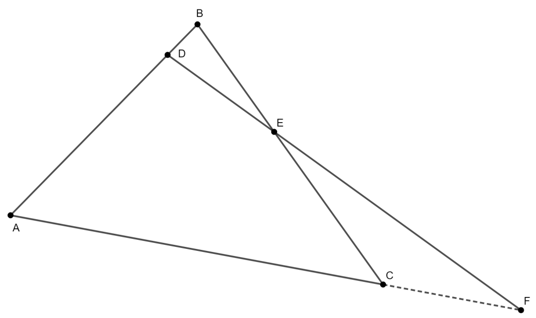
$$\frac{CF}{FB} = \frac{[COA]}{[AOB]}, \quad \frac{BD}{DA} = \frac{[BOC]}{[COA]}$$

$$\therefore \boxed{\frac{AE}{EC} \times \frac{CF}{FB} \times \frac{BD}{DA} = 1}$$

□

2.4 Menelaus' Theorem

This theorem addresses **collinearity**.



[1]Note: if $\frac{a}{b} = \frac{c}{d} \implies \frac{a}{b} = \frac{a-c}{b-d}$, by considering $\frac{a}{b} = \frac{a(b-d)}{b(b-d)}$.

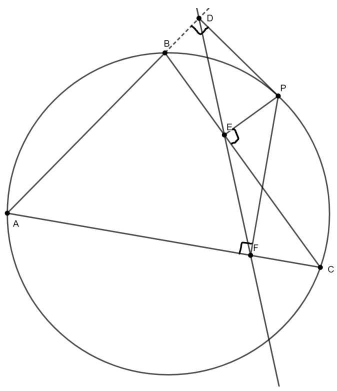
Drop perpendiculars ('orthogonal projections') from A, B, C to line DEF to form points A', B', C' respectively. By similar triangles

$$\frac{AD}{DB} = \frac{AA'}{BB'}, \ \frac{BE}{EC} = \frac{BB'}{CC'}, \ \frac{CF}{FA} = \frac{CC'}{AA'}$$

$$\therefore \boxed{\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1}$$

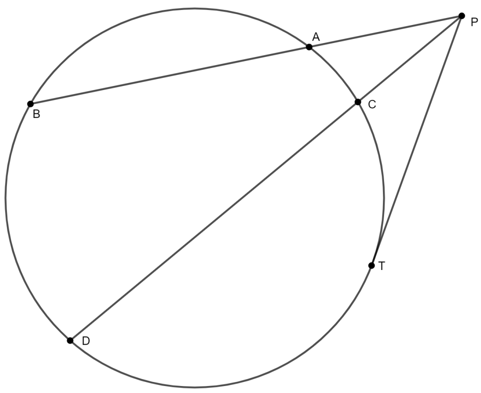
2.5 Simson Line

A very interesting and neat result.[2] For a point P on the circle, the three closest points to P on lines AB, BC and AC are collinear. Please note that the proof has been left out.



Points D, E, F are collinear.

2.6 Power of a Point



$\triangle PAD$ and $\triangle PCB$ are indirectly similar. It follows that

$$PA \times PB = PC \times CD$$

Also, $\triangle PAT$ is indirectly similar to $\triangle PTB$ (using the alternate segment theorem), so

$$PA \times PB = PT^2$$

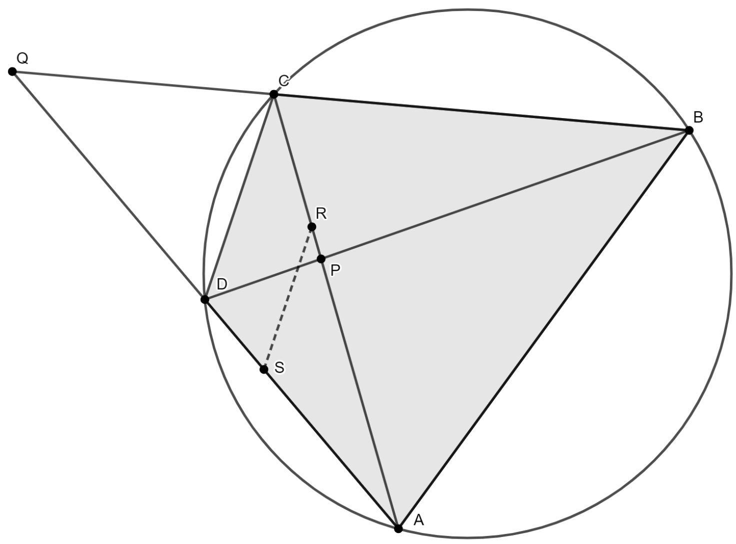
[2]This completely unlocks BMO1 2015 Q5, if you happen to care.

3 Example Problem 2

This is **problem 3** from *BMO2 2017*.

Consider a cyclic quadrilateral $ABCD$. The diagonals AC and BD meet at P, and the rays AD and BC meet at Q. The internal angle bisector of $\angle BQA$ meets AC at R and the internal angle bisector of $\angle APD$ meets AD at S. Prove that RS is parallel to CD.

Solution. Clearly, from the mentions of angle bisectors in the question, angle bisector theorem must be at play.



By the angle bisector theorem:

$$\frac{AP}{DP} = \frac{AS}{DS} \quad (\triangle DAP) \tag{1}$$
$$\frac{QA}{QC} = \frac{AR}{CR} \quad (\triangle QAC) \tag{2}$$

By the intercept theorem, if $RS \parallel CD \iff \frac{AS}{SD} = \frac{AR}{CR}$ $(\triangle DAC)$. So must show (1) = (2). Focus on the L.H.S.

$$\triangle PDA \sim \triangle PBC \implies \frac{AP}{DP} = \frac{BP}{CP}$$

Power of a point states $QC \times QB = QD \times QA \implies \frac{QA}{QC} = \frac{QB}{QD}$.

Let $\angle BDA = \alpha = \angle BCA$, $\angle CDB = \beta$. By sine rule in $\triangle QDB$,

$$\frac{QB}{\sin(180^\circ - \alpha)} = \frac{QD}{\sin\beta} \implies \frac{QB}{QD} = \frac{\sin\alpha}{\sin\beta}$$

In $\triangle CPB$,

$$\frac{BP}{\sin\alpha} = \frac{CP}{\sin\beta} \implies \frac{BP}{CP} = \frac{\sin\alpha}{\sin\beta}$$

$$\therefore \frac{QB}{QD} = \frac{BP}{CP} = \frac{QA}{QC} = \frac{AP}{DP} \implies \frac{AS}{DS} = \frac{AR}{CR}$$

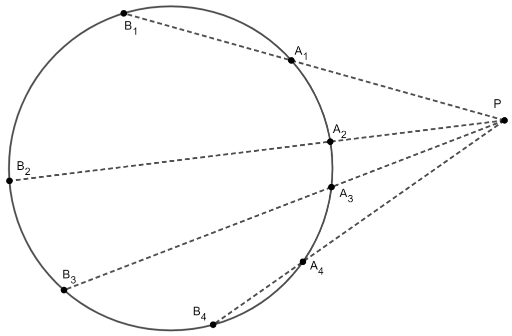
□

4 Further complex theorems (less relevant)

Here are some really interesting, and only slightly more complex, ideas in geometry. These are not explicitly required as theorems to recall for solving problems in many competitions, though help in gaining a better command of Euclidean geometry overall.

4.1 Generalising Power of a Point

These three results about tangents, secants and chords are unified by the idea that the quantity $PA \times PB$ depends on the relative positions of the point P and the circle, but does **not** on which chord, secant or tangent is used to define A and B.

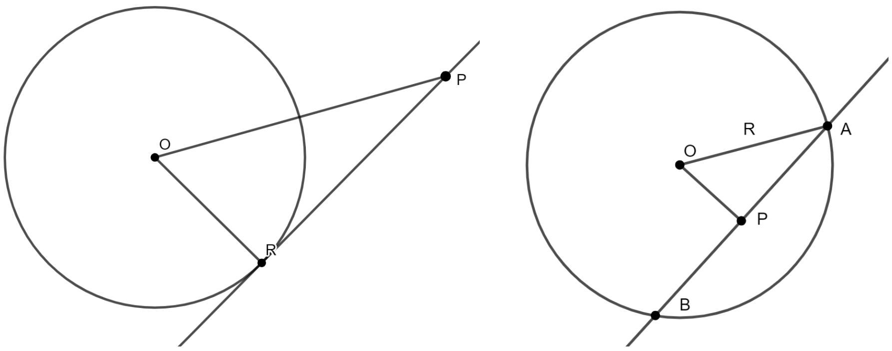


Prompted by this, we refer $PA \times PB$ as the *power* of P with respect to the circle. It is helpful to make this a signed quantity. This can be done by assigning an arbitrary direction to the line through P, and then regarding PA and PB as directed or **signed** lengths.

Thus,

1. if P is outside the circle, it will have positive power.
2. if P is inside the circle, PA and PB are in opposite directions so the power of P is negative.
3. the power of a point P on the circle is 0.

Let the circle have centre O and radius R. If P is outside the circle, the power of P is PT^2 and by Pythagoras, this is $OP^2 - R^2$.



Exactly the same formula is valid for P inside the circle.

If you are familiar with vector notation, then a neat way to define the power of P with respect to the circle is **PA.PB**.[3]

You'll be familiar with the Cartesian equation of a circle with centre (a, b) and radius r.

$$(x-a)^2 + (y-b)^2 = r^2$$

which expands to

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

Now, the power of a point $P = (x, y)$ defined by $PO^2 - R^2$ is $(x-a)^2 + (y-b)^2 - r^2$ or equivalently $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2$.

Now suppose you have two circles with different centres (a_1, b_1) and (a_2, b_2) and radii r_1 and r_2. The power of P with respect to **both** these circle will be equal precisely when

$$(x-a_1)^2 + (y-b_1)^2 - r_1^2 = (x-a_2)^2 + (y-b_2)^2 - r_2^2$$

which can be rewritten to

$$2(a_1 - a_2)x + 2(b_1 - b_2)y + k = 0$$

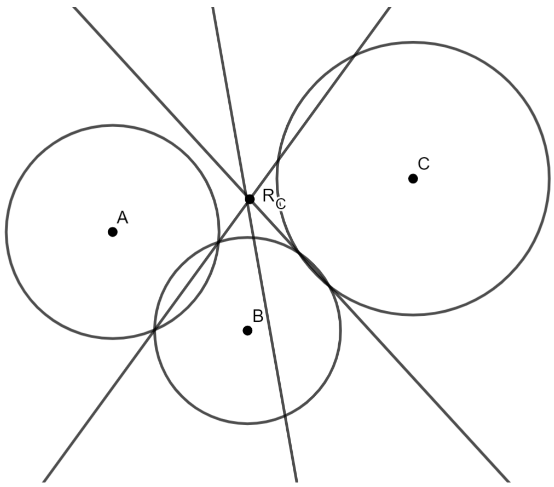
Now that's brilliant! This is the equation of a line known as the *radical axis* of the two circles.

4.2 Radical Axes

Given two circles, the radical axis of the two circles is the set of points with equal power with respect to both the circles.

The radical axis is **perpendicular** to the line of centres.

1. If the two circles intersect, then their radical axis is the straight line through their points of intersection.
2. If the two circles are tangent, then their radical axis is the line tangent to both circles at their point of tangency.
3. If the two circles do not intersect, then their radical axis is the line directly in between them.

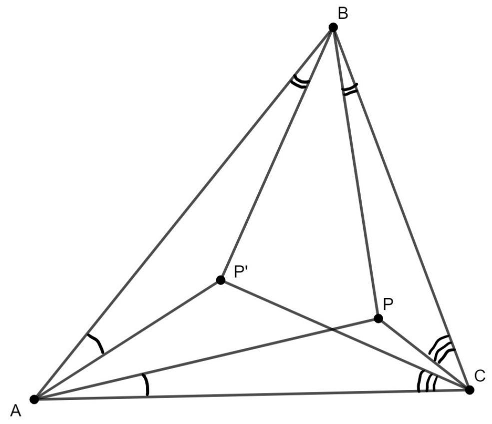


For three circles with centres which are not collinear, then the radical axes meet at a unique point (**concurrent**) and they specifically meet at the *radical centre*.

[3]This quantity is positive when both vectors point the same way and negative when they are pointing in opposite directions.

4.3 Isogonal Conjugate

The term *isogonal* means *'to have similar angles'*. Consider triangle ABC and P a point in the plane that does **not** lie on one of the sides of ABC. The line AP can be reflected in the internal or external angle bisector at A. Similar treatment can be done at vertex B and C and these newly reflected lines meet at a point P', the *isogonal conjugate* of P.



The incentre and three excentres of ABC are the isogonal conjugates of themselves.

4.4 Nine Point Circle

This is just in general a really cool fact, so I thought to end the theory side of things with it. It's rare in a setup that we find *nine* concyclic points! These nine points are:

- Midpoint of each side.
- Perpendiculars from opposite vertices.
- Midpoint of line from each vertex to orthocentre.